American Journal of Mathematical and Computer Applications

Research Article
Volume 1 | Issue 1

Spin and Representations

Wonmyeong Cho

Department of Mathematical Sciences, Seoul National University, South Korea.

***Corresponding author:** Wonmyeong Cho, Department of Mathematical Sciences, Seoul National University, South Korea.

Submitted: 10 Jan 2025 **Accepted:** 20 Jan 2025 **Published:** 25 Jan 2025

***Citation:** : Cho, W (2025) Spin and Representations. American j of math and comput applications. 1(1), 01-11.*

Abstarct

We derive the representation theory of SU (2) from the expository theory of Lie groups and Lie algebras. Based on this, the mathematics of non-relativistic quantum mechanics of a spin 1 particle are described from a representation-theoretic perspective, and are extended to many particle systems.

Lie Groups and Lie Algebras

We begin by evaluating the Lie algebras for some matrix groups, which will give us an easier method to construct the desired representations at hand.

Definition

A Lie group G is a topological group that is also a manifold, such that the group operation G × G → G is differentiable. The Lie algebra of G is defined to be the tangent space of G at the identity element e, and is denoted as g.

In physics, we are mainly interested in the matrix groups U (n), SO(n), SU (n) as representations of the group of symmetries of a certain object. For example, SO(3) represents the group of 3-dimensional rotations. Since the Lie algebra is a linearization of the Lie group, it is much easier to study than say, the corresponding neighborhood of the identity that we approximated.

Throughout, we assume G is a finite-dimensional matrix group, hence multiplication in g is defined as matrix multiplication. This, however, is not the operation that makes g into an algebra, and in general, g need not be closed under matrix multiplication.

Lemma 1.1

Let X ∈ g, ϵ ∈ R. For sufficiently small ϵ, there exists a group element of the form

$$1 + \epsilon X + \sum_{k=2}^{\infty} c_k \epsilon^k X^k$$

where ck are arbitrary real coefficients.

Notation

By use of the big-O notation, denote such element as 1 + ϵX + O(ϵ2).

Proposition 1.2

Let g ∈ G, X, Y ∈ g, t ∈ R. Then,

(i) $e^{tX} \in G$

(ii) $XY - YX \in g$

Proof

Observe that g is a real vector space, thus it suffices to prove (i) for X.

For sufficiently large $N \in \mathbb{N}$, there exists a group element of the form $1 + \frac{1}{N}X + \mathcal{O}(\frac{1}{N^2})$;

take the limit $N \to \infty$.

For (ii), define $\pi g(t) = ge^{tX}g{-}1$. The tangent vector at

$$\frac{d}{dt}\pi_g(t)\Big|_{t=0} = gXg^{-1} \in \mathfrak{g}$$

Take $g = e^{tY}$, then $e^{tY}Xe^{-tY} \in g$; differentiate at $t = 0$ to obtain the desired result.

Definition

We define the commutator of X, Y ∈ g as $[X, Y] = XY - YX$.

We are now ready to compute the Lie algebras of $U(n)$ and $SU(n)$.

Lemma 1.3

As vector spaces, dim G = dim g.

This follows from a basic result in differential topology that tangent spaces of a manifold have the same dimension as the manifold itself. A proof is in [GP], page 9.

Theorem 1.4

The Lie algebras of U (n) and SU (n) are given as follows:

$u(n) = \{X \in M_C(n) \mid X + X^\dagger = 0\}$

$su(n) = \{X \in \mathrm{M}_C(n) \mid X + X^\dagger = 0, \mathrm{tr}\, X = 0\}$

Proof

We start by showing that dim $U(n) = n^2$.

The constraint $UU^\dagger = 1$ reads $\sum_k u_{ik}\bar{u}_{jk} = \delta_{ij}$;

i = j yields 1 constraint in R, and i ≠ j yields 1 constraint in R, I, each. Thus, dim U (n) = $2n^2 - n - 2 \times \frac{n(n-1)}{2} = n^2$

Next, given $X \in u(n)$, $t \in \mathbb{R}$, we have $e^{tX}(e^{tX})^\dagger = e^{tX}e^{tX^\dagger} = 1$.

Differentiating at $t = 0$ yields $X + X^\dagger = 0$. Define $C = \{X \in MC(n) \mid X + X^\dagger = 0\}$. Counting the degrees of freedom of X, we deduce dim $C = n^2$, thus $C = u(n)$.

For su(n), we use the fact that SU (n) = U (n) ∩ SL(n) thus su(n) = u(n) ∩ sl(n).

Given $X \in sl(n)$, $\det(e^X) = e^{\mathrm{tr}\, X} = 1$, thus tr $X = 0$.

The space of traceless matrices and SL(n) both have dimension $2n^2{-}2$; we are done.

SU(2) and Spin Representations

We are particularly interested in SU(2) in non-relativistic quantum mechanics as the symmetric group of the Hilbert space of a particle carrying spin. What is spin, you ask? For now, I'll say that it is a special kind of angular momentum that is an instrinsic property of the particle. In that sense, we expect SU(2) to behave similarly to the group of rotations, SO(3). By deriving the fundamental commutator relations for the spin operators, we can finally begin to dig into the theory of spin. Let's get into it!

First though, to be able to draw this connection between quantum mechanics and representations, we need to introduce some basic physical language.

Definition

The wavefunction of a particle is any function that satiesfies the equation

Hˆ|ψ⟩ = E|ψ⟩.

This eigenvalue equation is called the Time-independent Schr¨odinger equation (TISE for short) and is determined by the Hamiltonian operator Hˆ of the particle. The eigenvalues E of the equation are called the energies of the particle. Any operator that commutes with the Hamiltonian is said to be compatible. The associated eigenspace of the equation is called the Hilbert space of the particle.

In essence, what we are trying to do is, instead of solving the Schr¨odinger equation directly (which is, for most of the time, near impossible), try to find some set of com- patible operators that give us information on what the Hilbert space looks like. We have no interest in what form the wavefunctions take; they can be functions, they can be vectors, they can be cats and dogs, as long as they spit out an energy eigenvalue.

Just remember this; anything we do here has an analogue in the TISE picture, and solving the TISE is equivalent to knowing the action of the Hamiltonian on the Hilbert space. Physicists seem to go to far lengths to solve that innocent-looking eigenvalue equation for some random particle whose existence is not even known; what we are about to do now is one of the cleaner methods of doing so.

From Theorem 1.4, we see that su(2) is generated by

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, X_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, X_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Definition

The Pauli matrices are σj = 2iXj, and the spin operators are

$S_j = iX_j = \frac{1}{2}\sigma_j$, for j = 1, 2, 3; the ladder operators are $S\pm = S_1 \pm iS_2$.

The spin operators satisfy the following commutation relations:

$$[S_i, S_j] = i\epsilon_{ijk}S_k \quad [S_3, S_\pm] = \pm S_\pm \quad [S_+, S_-] = 2S_3$$

Definition

The complexification of a Lie algebra g spanned by basis B over R is

defined as the space spanned by B over C, and is denoted as gC.

su(2)C = spanC $\{X_1, X_2, X_3\}$ = spanR $\{X_1, X_2, X_3, S_1, S_2, S_3\}$ = spanC $\{S+, S-, S_3\}$

Definition. Let g be a Lie algebra and V a vector space. A Lie algebra representa- tion is a map Φ :

g → gl(V) between Lie algebras, such that, for all a, b ∈ R, X, Y ∈ g:

(i) Φ(aX + bY) = aΦ(X) + bΦ(Y)

(ii) Φ([X, Y]) = [Φ(X), Φ(Y)]

Before putting this definition to use, we first classify the irreps of U (1).

Lemma 2.1

(Quantization Condition). Every irrep of U (1) can be written as

ϕn : u '→ un, where n is an integer.

Proof

Elements of U (1) are simply complex numbers such that uu¯ = 1, i.e. u = eiθ. Since U (1) is abelian, all its irreps have dimension 1, by Schur's lemma.

Differentiating ϕ in terms of θ, we have

$$\frac{d}{d\theta}\phi(e^{i\theta}) = \lim_{\epsilon\to 0}\frac{\phi(e^{i(\theta+\epsilon)}) - \phi(e^{i\theta})}{\epsilon} = \phi(e^{i\theta})\lim_{\epsilon\to 0}\frac{\phi(e^{i\epsilon}) - \phi(1)}{\epsilon} = n\phi(e^{i\theta})$$

where n = $\phi'(0)$; thus $\phi(e^{i\theta}) = e^{in\theta}$, and n is bounded by the homomorphism condition

e2πin = 1. The result follows.

It must be commented on that it is not immediately obvious how this connects to the quantization derived from, say, boundary conditions of a wave function. Well, what exactly do the boundary conditions signify? They are manifestations of the conditions that the representation of the associated Hamiltonian in Hilbert space must satisfy. In other words, the periodicity of the wave function correlates directly to the cyclicity of U (1). We demonstrate this by example of the spin operator that lives in SU (2).

Before doing so, we introduce a useful tool that allows us to go back and forth from Lie groups and Lie algebras.

Theorem 2.2

Let G be a simply connected Lie group with Lie algebra g.

(i) Let $\phi : G \to GL(V)$ be a Lie group representation. Then, ϕ induces a Lie algebra representation Φ with the property that $\phi(eX) = e\Phi(X)$ for all $X \in g$, given by

$$\Phi(X) = \frac{d}{dt}\phi(e^{tX})\Big|_{t=0}$$

(ii) Every Lie algebra representation $\Phi : g \to gl(V)$ arises in this manner.

The proof is left to [H], page 60, Theorem 3.28 for (i) and page 119, Theorem 5.6 for (ii). Since SU (2) is homeomorphic to the 3-sphere (write out the constraints for $U^\dagger U = UU^\dagger = 1$), it is simply connected and thus we may apply the above result.

Theorem 2.3

(The Spin Representation). For every nonnegative half-integer s, there is a unique irrep of SU (2) of dimension 2s + 1, which induces a representation of su(2) of the same dimension, given by

$$\Phi_s(S_3) = \begin{pmatrix} s & 0 & \dots & 0 \\ 0 & s-1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -s \end{pmatrix},$$

$$\Phi_s(S_+) = \begin{pmatrix} 0 & b_{s-1} & 0 & \dots & 0 \\ 0 & 0 & b_{s-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{-s} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \Phi_s(S_-) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ b_{s-1} & 0 & \dots & 0 & 0 \\ 0 & b_{s-2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{-s} & 0 \end{pmatrix}$$

where $b_m = \sqrt{s(s+1) - m(m+1)}$, for m = −s, . . . , s − 1 are chosen for convenience.

Proof. Let ϕ : SU (2) → GL(V) be an irrep with finite dimension.

We begin with S_3; notice that $e^{2i\theta S3} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$.

thus the subgroup it generates in SU (2) is isomorphic to U (1).

Choose a basis where ϕ(e2iθS3) is diagonal, and using Theorem 2.2(i), we see that Φ(S3) is a diagonal matrix consisting of half integers.

Label the diagonal entries ai, where i = 0, 1, . . . , n in non-decreasing order. Here's the catch: given an eigenvector xi of Φ(S3) with eigenvalue ai, we have

$$\Phi(S_3)(\Phi(S_+)x_i) = (a_i + 1)(\Phi(S_+)x_i), \Phi(S_3)(\Phi(S_-)x_i) = (a_i - 1)(\Phi(S_-)x_i)$$

$$[\Phi(S_3), \Phi(S_+)]x_i + \Phi(S_+)\Phi(S_3)x_i = \Phi(S_3)\Phi(S_+)x_i = (1 + a_i)\Phi(S_+)x_i$$

The calculation is similar for $\Phi(S_-)$. This tells us that, since the set of eigenvalues $\{a_i\}$

is bounded by $[a_0, a_n]$, $\Phi(S+)x_n$ and $\Phi(S-)x_0$ are zero vectors.

Claim: V is spanned by $B+ = \{x_0, \Phi(S+)(x_0), (\Phi(S+))^2(x_0), \dots\}$.

Note that the set is finite since it terminates eventually. It suffices to show that the space spanned by B+ is closed under action of su(2)C = spanC $\{S+, S-, S_3\}$, for then it follows that span B+ is a SU (2)-linear subspace of V , thus equals V .

Here, we use the fact that every U ∈ SU (2) is of the form $U = e^X$ for some X ∈ su(2).

The case is closed for $\Phi(S_3)$ and $\Phi(S_+)$. For $\Phi(S_-)$, induct on the power of Φ(S+): Φ(S−)(x0) = 0 for the base case; for the general case, observe

$\Phi(S_-)\Phi(S_+) = \Phi(S_+)\Phi(S_-) - [\Phi(S_+), \Phi(S_-)] = \Phi(S_+)\Phi(S_-) - 2\Phi(S_3)$

Apply both sides to $(\Phi(S_+))^{m-1}(x_0)$ to obtain the desired result.

The claim allows us to write $a_i = a_0 + i$. Now, consider the case n = 0.

Here, x0 = xn, thus V = span {x0}, and we have $\Phi(S_+) = \Phi(S_-) = \Phi(S_3) = 0$. It follows that ϕ is trivial, since ϕ(eX) = eΦ(X) = 1 for all X ∈ g.

Let V have dimension n + 1 for any positive integer n.

Composition of ϕ with the determinant representation D : GL(V) → GL1(C) ≃ C given by X '→ det X yields a 1-dimensional representation D ◦ ϕ : SU (2) → GL1(C).

This representation is irreducible, being of dimension 1, thus is trivial. Since det ϕ(eX) = etr Φ(X) = 1, we see that Φ(X) is traceless for all X ∈ g.

n

In particular, $\sum_{i=0}^{n} a_i = 0$ thus $a_i = -a_{n-i}$, which gives us S3.

Constructing S± from the eigenvector condition, we are done.

From the above proof, notice that $a_n = \frac{n}{2}$ can be any nonnegative half integer, and the dimension of the representation is given by 2an + 1 (labeled ‘s’ in the statement). The factors bm are arbitrary constants that melt into the eigenvector; they are chosen such that the theory of spin mimics that of angular momentum.

Definition

From the statement of Theorem 2.3, this Lie algebra representation char- acterizes the Hilbert space of a particle with spin s, and thus is called the is called the spin s representation. The eigenvectors xi are called states, and we denote them as bras ⟨sm| and kets |sm⟩ for column and row vectors, respectively, where m is the associated eigenvalue of Φs(S3) and is called the magnetic quantum number.

An observable is a Hermitian operator that has V as its eigenspace. It is called so because it has real eigenvalues and thus will have a physical manifestation. Any wave function associated to an observable compatible with the Hamiltonian of this particle will be a linear combination of these states.

We now have that Φ(S+)|sm⟩ ∝ |s(m + 1)⟩, Φ(S−)|sm⟩ ∝ |s(m − 1)⟩.

Finally, we define a new observable, called the spin-squared operator.

$$S^2 := S_1^2 + S_2^2 + S_3^2 = S_+S_- - S_3 + S_3^2$$

Notice that it shares simultaneous eigenvectors as S3, hence $[S^2, S_3] = 0$.

We summarize our results below:

Theorem 2.4. The action of the spin operators on the states are given as follows:

$$S_3|sm\rangle = m|sm\rangle, S^2|sm\rangle = s(s+1)|sm\rangle$$

$$S_\pm|sm\rangle = \sqrt{s(s+1) - m(m\pm 1)}|s(m\pm 1)\rangle$$

3. Many Particle Systems and Clebsch-Gordan Coefficients

We are now ready to construct the theory for many particle systems. First, consider what space is spanned by the states of a two-particle system, each with spin s1, s2. Each pair of two individual state |s1m1⟩|s2m2⟩ is a state of the system, thus our space has as basis all such states; thus naturally, we construct the two-particle Hilbert space as the tensor product of the two single particle Hilbert spaces.

The generalization to any system consisting of a finite number of particles is straight- forward. To avoid messy notation and ellipses, I will proceed with two-particle systems, but keep in mind that all these concepts generalize to any finite number of particles.

Definition

Let a system X be composed of particles 1, 2 with spin s1, s2, respectively. We denote the spin operators of each system by S(1), S(2)(i = 1, 2, 3, ±), respectively.

The total spin operators are defined as Si = S(1) ⊗ 1 + 1 ⊗ S(2)(i = 1, 2, 3, ±),

and the total spin-squared operator is then calculated as

$$S^2 = S_1^2 + S_2^2 + S_3^2 = S^{2^{(1)}} \otimes 1 + 1 \otimes S^{2^{(2)}} + 2(S_1^{(1)} \otimes S_1^{(2)} + S_2^{(1)} \otimes S_2^{(2)} + S_3^{(1)} \otimes S_3^{(2)})$$

Proposition 3.1. The total spin operators satisfy the commutation relations

[Si, Sj] = iϵijkSk [S3, S±] = ±S± [S+, S−] = 2S3

and therefore induce a basis of states |sm⟩, where s runs from |s1 − s2| to s1 + s2.

Proof. The commutation relations follow from simple algebra. We desire to find simultaneous eigenstates of S^2 and S_3.

Observe that for given state $|s_1m_1\rangle|s_2m_2\rangle$, we have

$$S_3|s_1m_1\rangle|s_2m_2\rangle = (m_1+m_2)|s_1m_1\rangle|s_2m_2\rangle$$

thus we have $m = m_1 + m_2$, and it suffices to consider the degeneracy of the associated eigenspace for each $m = -(s_1 + s_2), \ldots, s_1 + s_2$.

Assume $s_1 \geq s_2$ wlog; the number of ordered pairs (m_1, m_2) summing to m with the constraints $|m1| < s_1$, $|m_2| < s_2$ gives the degeneracy level.

Counting the pairs, we see that the degeneracy level is 2s2 + 1 for |m| ≤ s1 − s2, and decreases by 1 as |m| increases by 1 for each s1 − s2, . . . , s1 + s2.

Since the states are independent and the total number is

$$(2(s_1 - s_2) + 1)(2s_2 + 1) + 2\sum_{n=1}^{2s_2} n = (2s_1 + 1)(2s_2 + 1),$$

these states fill the whole of the Hilbert space spanned by {|s1m1⟩|s2m2⟩}.

From the above proof, observe that for each s = |s1 − s2|, . . . , s1 + s2, the total spin operator induces a spin s representation. By Theorem 2.3, these are irreps of SU (2), thus we have effectively decomposed the tensor product of two irreps of SU (2).

Since this is the main result in terms of representation theory, we formally state it below:

Theorem 3.2. Let Vs1 , Vs2 be the SU (2)-linear spaces associated to the spin s1, s2

representations, respectively. Then, the decomposition of Vs1 ⊗ Vs2 is given by

$$V_{s_1} \otimes V_{s_2} = \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} V_s$$

where Vs is the SU (2)-linear space associated to the spin s representation.

Now that we have done the two-particle case, having obtained a new basis in the form |sm⟩, we may view the tensor product Hilbert space as a direct sum of familiar ones; this allows us to apply the construction of total spin operators inductively.

We conclude with a discussion of the transformation of the |sm⟩ basis to the $|s_1m_1\rangle|s_2m_2\rangle$ basis. This topic is of great significance in physics since it gives a relation between the microscopic and macroscopic scales.

Definition

We define the Clebsch-Gordan coefficients $C^{s_1 s_2 s}_{m_1 m_2 m}$ as follows:

$$|sm\rangle = \sum_{(m_1,m_2)} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 m_1\rangle |s_2 m_2\rangle$$

Note that $C^{s_1 s_2 s}_{m_1 m_2 m} = 0$ unless $m_1 + m_2 = m$, considering action by S_3.

I will guide you through an elementary example of calculating these coefficients, then tell you the general method.

Example

Take two spin $\frac{1}{2}$ particles; denote $|\frac{1}{2}\frac{1}{2}\rangle = |\uparrow\rangle, |\frac{1}{2}\text{-}\frac{1}{2}\rangle = |\downarrow\rangle$.

These are the so-called spin 'up' and 'down' states, applicable to any spin 1 particle

(in particular, all known fermions).

We desire to find the transformation $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \rightarrow \{|11\rangle, |10\rangle, |1-1\rangle, |00\rangle\}$. Starting with the top state of highest spin, we have $|11\rangle = |\uparrow\uparrow\rangle$.

$$S_-|11\rangle = \sqrt{2}|10\rangle = (S_-^{(1)} \otimes 1 + 1 \otimes S_-^{(2)})|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle$$

$$S_-|10\rangle = \sqrt{2}|1-1\rangle = (S_-^{(1)} \otimes 1 + 1 \otimes S_-^{(2)})\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \sqrt{2}|\downarrow\downarrow\rangle$$

Solve for $|00\rangle$ using the orthogonality condition $\langle 10|00\rangle = 0$. Conventionally, we choose the sign so that the state with the highest m1 (assuming $s_1 \geq s_2$) is positive.

$$|11\rangle = |\uparrow\uparrow\rangle, |10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |1-1\rangle = |\downarrow\downarrow\rangle, |00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

For the general case, start with the topmost state $|(s_1 + s_2)(s_1 + s_2)\rangle = |s_1 s_1\rangle|s_2 s_2\rangle$, as with the example. Climb down the ladder until you reach the bottom rung; symmetry of the ladder indicates that the coefficients are symmetric about the m = 0 rung:

$$\langle s(m-1)|S_-|sm\rangle = \langle s(-m+1)|S_+|s-m\rangle$$

which saves you half the calculations. Moving onto the total spin $s_1 + s_2 - 1$ states, use the orthogonality condition on $|(s_1 + s_2 - 1)(s_1 + s_2 - 1)\rangle$ and normalize, putting the plus sign on the 'upper' state. Climb down the ladder once more to retreive all the coefficients for this spin. For each time you step down a spin, you get one more orthogonality condition to work with; rinse and repeat.

The general Clebsch-Gordan coefficients can be derived in explicit form using this method, since we already know our starting point at the topmost rung. It is not a pleasing calculation to work out, but if you want to see the general form, refer to [B], page 171, and the calculations leading up to (2.41).

Here is a particular general case of this form, where we have a system consisting of a particle with arbitrary spin and a particle with spin $\frac{1}{2}$.

Theorem 3.3

Let s_1 be any half-integer, $s_2 = \frac{1}{2}$. Then, $s = s_1 \pm \frac{1}{2}$, and

$$|sm\rangle = \sqrt{\frac{s_1 \pm m + 1/2}{2s_1+1}}|s_1(m-\tfrac{1}{2})\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle \pm \sqrt{\frac{s_1 \mp m + 1/2}{2s_1+1}}|s_1(m+\tfrac{1}{2})\rangle|\tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle.$$

Proof

Have $|sm\rangle = A|s_1(m-\frac{1}{2})\rangle|\frac{1}{2}\frac{1}{2}\rangle + B|s_1(m+\frac{1}{2})\rangle|\frac{1}{2}\text{-}\frac{1}{2}\rangle$..

Write $S^2 = S_+S_- + S_3^2 - S_3 = S_-S_+ + S_3^2 + S_3$.

We desire to find A, B such that $|sm\rangle$ is an eigenstate of S^2 with eigenvalue $s(s+1)$.

Setting $x = (s_1 + \frac{1}{2})^2, y = \sqrt{x - m^2}$ for convenience, we have

$$S^2|s_1(m-\tfrac{1}{2})\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = (x+m)|s_1(m-\tfrac{1}{2})\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + y|s_1(m+\tfrac{1}{2})\rangle|\tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle$$

$$S^2|s_1(m+\tfrac{1}{2})\rangle|\tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle = (x-m)|s_1(m+\tfrac{1}{2})\rangle|\tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle + y|s_1(m-\tfrac{1}{2})\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$S^2|sm\rangle = s(s+1)|sm\rangle = ((x+m)A+yB)|s(m-\tfrac{1}{2})\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + (yA+(x-m)B)|s(m+\tfrac{1}{2})\rangle|\tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle$$

$$(x+m)A + yB = s(s+1)A, \;\; yA + (x-m)B = s(s+1)B$$

The two conditions should be one and the same; with some more algebra, we have

$$(x - s(s+1))^2 - x = 0, \text{ or } s = s_1 \pm \tfrac{1}{2}$$

All that is left to do is normalize and set the sign of A to be positive.

We end on a physical note by introducing the concept of entanglement.

Definition

A linear combination of states is said to be entangled if it cannot be written as a single tensor product of states.

What this means in our physical context is that, once we obtain a certain result for particle A (i.e. we ‘observe’ it), the state of particle B has been determined, regardless of any external conditions concerning the two particle system. The two particles cannot exist independently of each other, hence the term ‘entanglement’. Let this sink in for a second. If Alice observes particle A on the sun, then Bob sees particle B in a fixed state on the Earth from the very time frame that Alice observed particle A.

How can this be? The system cannot send information faster than light, but somehow the particles ‘knows’ instantly what has happened to its partner. This phenomenon is the basis for the famous EPR paradox, and arose a feverous discussion on causality of events and hidden variable theory, but that is beyond what I can hope to cover here.

Let's get back on track. I'll give you a simple example.

Example

Recall the system in the previous example consisting of two spin 1 particles.

It is easy to see that the states with m = 0 are entangled:

$$|10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),\ |00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

We already know that the tensor product must be of in a form such that m1 + m2 = m, but the forms written above are the exact linear combination of such states.

Suppose we measure the state |10⟩ with $S_3^{(1)}$, which returns m_1:

we obtain $\frac{1}{2}$ with probability $\frac{1}{2}$ and -$\frac{1}{2}$ with probability $\frac{1}{2}$.

Then, for each such measurement on particle 1, we know m2 without measuring it!

For any wavefunction consisting of a linear combination of states |sm⟩, by using the Clebsch-Gordan coefficients, we can decompose all of them into linearly independent basis states of the form $|s_1m_1\rangle|s_2m_2\rangle$. By measuring these states accordingly with the spin operators in the Hilbert space of each particle, we may analyze the entanglement problem associated with the original state.

We refer to measuring such a state as 'collapsing the wave function': once we restrict the component $|s_1m_1\rangle$, we are simply left with a linear combination of $|s_2m_2\rangle$ states, and the wavefunction is no longer entangled.

References

1. Differential Topology by Victor Guillemin and Alan Pollack, Prentice Hall, 1974
2. Lie Groups, Lie Algebras, and Representations by Brian Hall (2nd ed.), Springer, 2015
3. Representation Theory and Quantum Mechanics by Noah Miller, Harvard, 2018
4. Quantum Mechanics: Foundations and Applications by Arno Bohm (3rd ed.), Springer, 1993
5. Introduction to Quantum Mechanics by David Griffiths (2nd ed.), Prentice Hall, 2005
6. Quantum Mechanics by Claude Cohen-Tannoudji, Vol. 2, Wiley, 1977
7. Group Theoretical Aspects of Quantum Mechanics by Jos´e Alvarado, Leipzig, 2007