American Journal of Mathematical and Computer Applications

Research Article
Volume 1 | Issue 2

Free Ribbon Lemma for Surface-Link

Akio Kawauchi

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, Osaka, Japan.

***Corresponding author:** Akio Kawauchi, Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, Osaka, Japan.

Submitted: 01-May-2025 **Accepted:** 21-May-2025 **Published:** 21-July-2025

Citation: **Kawauchi, A. (2025). Free Ribbon Lemma for Surface-Link. American Journal of Mathematical and Computer Applications 1(1): 01-07.**

Abstarct

A free surface-link is a surface-link whose fundamental group is a free group not necessarily meridian-based. Free ribbon lemma says that every free sphere- link in the 4-sphere is a ribbon sphere-link. Four different proofs of Free ribbon lemma are explained. The first proof is done in an earlier paper. The second proof is done by showing that there is an O2-handle basis of a ribbon surface- link. The third proof is done by removing the commuter relations from a Wirtinger presentation of a free group, which a paper on another proof of Free ribbon lemma complements. The fourth proof is given by the special case of the proof of the result that every free surface-link is a ribbon surface-link which is a stabilization of a free ribbon sphere-link. As a consequence, it is shown that a surface-link is a sublink of a free surface-link if and only if it is a stabilization of a ribbon sphere-link.

Keywords: Free Ribbon Lemma, Free Surface-Link, Ribbon Sphere-Link, Stabilization.

Mathematics Subject Classification 2010: Primary 57N13; Secondary 57Q45.

1. Introduction

A surface link is a closed, possibly disconnected, oriented surface F smoothly embedded in the 4-sphere S^4, and it is called a surface knot if F is connected. If F consists of 2-spheres F_i (i = 1, 2, . . . , r), then F is called a sphere-link (or an S^2-link) of r components. It is shown that a surface-link F is a trivial surface-link (i.e., bounds disjoint handlebodies in S^4) if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is a meridian-based free group, [1-3]. A surface-link F is ribbon if F is obtained from a trivial S^2-link O in S^4 by surgery along a smoothly embedded disjoint 1-handle system h^O on O, [4-7]. A surface-link F in the 4-sphere S^4 is free if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is a (not necessarily meridian-based) free group. In this paper, four different proofs of the following Free ribbon lemma and its generalization to a general free surface-link are explained.

Free ribbon lemma

Every free S^2-link in S^4 is a ribbon S^2-link.

Free ribbon lemma leads to the following conjectures: Poincarè conjecture, [8-11]. J. H. C. Whitehead asphericity conjecture for aspherical 2-complex, [12-15]. Kervaire conjecture on group weight, [16-20]. The first proof is given [13]. For convenience, an outline of the first proof is explained here.

First proof of Free ribbon lemma. Let L_i (i = 1, 2, . . . , r) be the components of a free S^2-link L in S^4. By a base change of the free fundamental group $\pi_1(S^4 \setminus L, x_0)$, take a basis x_i (i = 1, 2, . . . , r) of $\pi_1(S^4 \setminus L, x_0)$ inducing a meridian basis of L in $H_1(S^4 \setminus L; Z)$, [20]. Let Y be the 4-manifold obtained from S^4 by surgery along L, which is diffeomorphic to the connected sum of r copies $S^1 \times S^3$ (i = 1, 2, . . . , r) of $S^1 \times S^3$, [13,21]. Under a canonical isomorphism $\pi_1(S^4 \setminus L, x_0) \to \pi_1(Y, x_0)$, the factors $S^1 \times p_i$ (i = 1, 2, . . . , r) of $S^1 \times S^3$ (i = 1, 2, . . . , r) with suitable paths to the base point X_0 represent the basis x_i (i = 1, 2, . . . , r). Let k_i (i = 1, 2, . . . , r) be the loop system in Y produced from the components L_i (i = 1, 2, . . . , r) by the surgery. By using the fact that any homotopy deformations of k_i (i = 1, 2, . . . , r) in Y do not change the link type of the surface-link L in S^4, the loop system k_i (i = 1, 2, . . . , r) is homotopically deformed in Y so that the surface-link L in S4 obtained from the deformed loop system ki (i = 1, 2, . . . , r) by back surgery is a ribbon surface-link in S4, completing the proof of Free ribbon lemma.

To explain the second and third proofs of Free ribbon lemma, the notion of an O2-handle basis of a surface-link is needed, [1,22]. An O2-handle on a surface- link F in S^4 is a pair (D × I, D′ × I) of 2-handles D × I, D′ × I on F in S^4 which intersect orthogonally only with the attaching parts (∂D) × I, (∂D′) × I to F, so that the intersection Q = (∂D) × I ∩ (∂D′) × I is a square. Let (D × I, D′ × I) be an O2-handle pair on a surface-link F. Let F(D × I) and F(D′ × I) be the surface-links obtained from F by the surgeries along D × I and D′ × I, respectively. Let F(D × I, D′ × I) be the surface-link which is the union δ ∪ F^c_δ of the plumbed disk

$$\delta = \delta_{D\times I, D'\times I} = D \times \partial I \cup Q \cup D' \times \partial I.$$

The surface-links F^c_δ F(D × I), F (D′ × I) and F (D × I, D′ × I) are equivalent surface- links, [1]. An O2-handle basis of a surface-link F is a disjoint system of O2-handle pairs (D_i × I, D′ × I) (i = 1, 2, . . . , r) on F in S^4 such that the boundary loop pair system (∂D_i, ∂D′) (i = 1, 2, . . . , r) of the core disk system (D_i, D'_i) (i = 1, 2, . . . , r) of (D_i × I, D'_i × I) (i = 1, 2, . . . , r) is a spin loop basis for F in S^4, which is a system of a spin loop basis of every component F_i of F. Note that there is a spin loop basis for every surface-knot in F, [3]. In this paper, for simplicity, an O2-handle basis (D_i × I, D′ × I) (i = 1, 2, . . . , r) for F is denoted by (D × I, D′ × I). The surgery surface-link of F by (D_i × I, D′ × I) (i = 1, 2, . . . , r) is denoted by F (D × I, D′ × I). The following theorem is shown for the second and third proofs of Free ribbon lemma.

Theorem 1.1

For every free ribbon surface-link F in S^4, there is an O2-handle basis (D × I, D′ × I) on F in S^4 such that D × I belongs to the 1-handle system of the ribbon surface-link F.

The second proof of Free ribbon lemma is explained as follows.

Second Proof of Free Ribbon Lemma

Let L be a free S^2-link. Then there is a ribbon surface-link F such that the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is isomorphic to the free fundamental group $\pi_1(S^4 \setminus L, x_0)$ by a meridian-preserving isomorphism [23]. By Theorem 1.1, the surgery surface-link L′ = F (D×I, D′×I) is a ribbon S^2-link, [1, 22]. Then there is a meridian-preserving isomorphism $\pi_1(S^4 \setminus L', x_0) \to \pi_1(S^4 \setminus L, x_0)$ on free groups, which implies that L′ is equivalent to L, [13,24]. Thus, L is a ribbon S^2-link, completing the proof of Free ribbon lemma.

The third proof of Free ribbon lemma is related to a Wirtinger presentation of a free group. A finite group presentation (x_1, x_2, . . . , x_n| R_1, R_2, . . . , R_m) is a Wirtinger presentation if $R_j = W_{jxsj} W_j^{-1} x_{tj}^{-1}$ for some indexes s_j, t_j in {1, 2, . . . , n} for every j (j = 1, 2, . . . , m). The relator R_j is a commutator relation if $x_{sj} = x_{tj}$. It is well-known that a Wirtinger presentation of a finitely presented group G with $H_1(G; Z) = Z^r$ is always equivalent (without changing the gerenating set) to a Wirtinger presentation P such that the Wirtinger presentation P′ obtained by removing all the commutator relations from P has deficiency r. Such a Wirtinger presentation P is called a normal Wirtinger presentation. The following corollary is obtained from Theorem 1.1.

Corollary 1.2

If a free group G of rank r has a normal Wirtinger presentation P, then G has the Wirtinger presentation P′ of deficiency r obtained from P by removing all the commutator relations.

Proof of Corollary 1.2 assuming Theorem 1.1

Let P = (x_1, x_2, . . . , x_n| R_1, R_2, . . . , R_m) be a normal Wirtinger presentation of a free group G of rank r such that the relators R_j $(n - r + 1 \le j \le m)$ are the commutator relations. Let O be a trivial S^2-link of n components in S^4 such that the meridian basis of the free fundamental group $\pi_1(S^4 \setminus O, x_0)$ are identified with x_i (i = 1, 2, . . . , n). Let h_j (1 ≤ j ≤ m) be the 1-handles on O indicated by the relators R_j (1 ≤ j ≤ m). By the van Kampen theorem, the ribbon surface-link F in S^4 obtained by surgery along h_j $(1 \le j \le m)$ has the normal Wirtinger presentation P of the fundamental group $\pi_1(S^4 \setminus F, x_0)$ with the meridian generators set {x_1, x_2, . . . , x_n}, [24,25]. Let L be the ribbon surface-link obtained from O by surgery along the 1-handles h_j (1 ≤ j ≤ n − r), which is a ribbon S^2-link of r components, The fundamental group $\pi_1(S^4 \setminus L, x_0)$ has the Wirtinger presentation P′ of deficiency r obtained from P by removing all the commutator relations. By Theorem 1.1, the 1-handles h_j (n − r + 1 ≤ j ≤ m) are trivial 1-handles on O, so that $\pi_1(S^4 \setminus L, x_0)$ is isomorphic to $\pi_1(S^4 \setminus F, x_0)$ by a meridian-preserving isomorphism. This completes the proof of Corollary 1.2 assuming Theorem 1.1.

The author has published a paper on another proof of Free ribbon lemma, which this paper complements, [23]. The third proof of Free ribbon lemma is nothing but the proof of the paper except for adding to it the assertion of Corollary 1.5 which was missing from it. For convenience, an outline of the third proof is explained here.

Third Proof of Free Ribbon Lemma

Let L be a free S^2-link of r components. Since the fundamental group $G = \pi_1(S^4 \setminus L, x_0)$ is a free group with $H_1(G; Z) = Z^r$ and $H_2(G; Z) = 0$, there is a normal Wirtinger presentation P of G whose generator set comes from meridians of L in S^4, [23,26]. Note that there is also another method to find such a normal Wirtinger presentation P using a normal form of L in S^4, [4,24,25,27]. Let L′ be a ribbon S^2-link given by the Wirtinger presentation P′ obtained from P by removing all the commutators. By Corollary 1.2, there is a meridian-preserving isomorphism $\pi_1(S^4 \setminus L', x_0) \to \pi_1(S^4 \setminus L, x_0)$, so that L′ is equivalent to L. Thus, L is a ribbon S^2-link, completing the proof of Free ribbon lemma.

The fourth proof of Free ribbon lemma is given by a direct proof of the following theorem.

Theorem 1.3

Every free surface-link F in S^4 is a ribbon surface-link in S^4.

Fourth proof of Free ribbon lemma

It is obtained by restricting F to every free S^2-link, completing the proof of Free ribbon lemma.

Thus, after the proofs of Theorems 1.1 and 1.3, there are four different proofs of Free ribbon lemma.

To generalize the free ribbon lemma to a free surface-link, the notion of a stabilization of a surface-link is needed, [1,22]. A stabilization of a surface-link L is a connected sum $F = L\#_{k=1}^{s} T_k$ of L and a system of trivial torus-knots T_k (k = 1, 2, . . . , s).

By granting s = 0, a surface-link L itself is regarded as a stabilization of L. Free ribbon lemma is generalized to a general free surface-link as follows.

Corollary 1.4. Every free surface-link F in S^4 is a stabilization of a free ribbon S^2-link L in S^4.

Proof of Crollary 1.4 assuming Theorems 1.1 and 1.3. Theorem 1.1 implies that every free surface-link F is a stabilization of a free S^2-link L, [1]. By Free ribbon lemma, the free S^2-link L is a ribbon S^2-link. This completes the proof of Crollary 1.4 assuming Theorems 1.1 and 1.3.

It is shown that an S^2-link L is a sublink of a free S^2-link if and only if L is a ribbon S^2-link, [13]. The following corollary generalizes this property to a general surface-link.

Corollary 1.5. A surface-link L in S^4 is a sublink of a free surface-link F in S^4 if and only if L is a stabilization of a ribbon S^2-link in S^4.

Proof of Corollary 1.5 assuming Theorem 1.3. If L is a sublink of a free surface-link F , then L is a stabilization of a ribbon S^2-link since every free surface-link is a stabilization of a free ribbon S^2-link by Corollary 1.2. Conversely, if L is a stabilization of a ribbon S^2-link, then L is a sublink of a stabilization of a free ribbon S^2-link which is a free surface-link F since every ribbon S^2-link is a sublink of a free S^2-link. This completes the proof of Corollary 1.5 assuming Theorem 1.3.

2. Proofs of Theorems 1.1 and 1.3.

Let F be a free surface-link in S^4 with components F_i (i = 1, 2, . . . , r). Let N(F) = $\cup_{i=1}^{r} N(F_i)$ be a tubular neighborhood of $F = \cup_{i=1}^{r} F_i$ F_i in S^4 which is a trivial normal disk bundle F × D^2 over F, where D^2 denotes the unit disk of complex numbers of norm ≦ 1. Let E = E(F) =cl(S^4 \ N(F)) be the exterior of F in S^4. The boundary ∂E = ∂N(F) = $\cup_{i=1}^{r} \partial N(F_i)$of the exterior E is a trivial normal circle bundle over F = $\cup_{i=1}^{r} F_i$. Identify ∂N(F_i) =F_i × S^1 for S^1 = ∂D^2 such that the composite inclusion F_i × 1 → ∂N(F_i) → cl(S^4 \ N(F_i)) induces the zero-map in the integral first homology. The following lemma uses the assumption that the fundamental group $\pi_1(E, x_0)$ is a free group of rank r and the fact that the first homology group H_1(E; Z) is a free abelian group of rank r with meridian basis.

Lemma 2.1

The composite inclusion F_i × 1 → ∂N(F_i) → E is null-homotopic for all i.

Proof of Lemma 2.1

Since ∂N(F_i) = F_i × S^1, the fundamental group elements between the factors F_i × 1 and q_i × S^1 are commutive. Let a_i (i = 1, 2, . . . , r) be embedded edges with common vertex x_0 in E such that a_i \ {x_0} (i = 1, 2, . . . , r) are mutually disjoint and $a_i \cap (\cup_{j=1}^{r} F_j \times 1)$ for a point p_i of F_i × 1. The surface F_i × 1 in ∂N(F_i) = F_i × S^1 is chosen so that the inclusion F_i × 1 → cl(S^4 \ N(F_i)) induces the zero-map in the integral first homology. Since H_1(E; Z) is a free abelian group of rank r with meridian basis and $\pi_1(E, x_0)$ is a free group of rank r, the image of the homomorphism $\pi_1(a_i \cup F_i \times S_1, x_0) \to \pi_1(E, x_0)$ is an infinite cyclic group generated by the homotopy class [a_i ∪ p_i × S_1]. This implies that the inclusion F_i × 1 → E is null-homotopic. This completes the proof of Lemma 2.1.

By using the free group $\pi_1(E, x_0)$ of rank r, let

$$\Gamma = ((\cup_{i=1}^{r} a_i) \cup (\cup_{i=1}^{r} C_i)$$

be a connected graph with a degree one vertex x_0 in the interior Int(E) of E consisting of embedded edges a_i (i = 1, 2, . . . , r) with the common base point x_0 and disjoint embedded circles C_i (i = 1, 2, . . . , r) such that

(1) the half-open edges a_i \ {x_0} (i = 1, 2, . . . , r) are mutually disjoint and a_i ∩ C_j) = v_i, a point in C_i for every i,

(2) The inclusion i : (Γ, x_0) → (E, x_0) induces an isomorphism $i_\#$: $\pi_1(K, x_0) \to \pi_1(E, x_0)$, and

(3) The homology class [p_i × S^1] = [C_i] in H_1(E; Z) for all i.

In fact, by (2), the homotopy classes [a_i ∪ C_i] (i = 1, 2, . . . , r) form a basis of the free group $\pi_1(E, q_0)$. (3) is obtained by a base change of the free group $\pi_1(E, x_0)$, [20]. Since Γ is a K(π, 1)-space, there is a piecewise-linear map f : (E, x_0) → (Γ, x_0) inducing the inverse isomorphism $f_\# = (i_\#)^{-1} : \pi_1(E, q_0) \to \pi_1(\Gamma, q_0)$, and by the homotopy extension property, the restriction of f to Γ is the identity map, [29]. The restriction of f to ∂E is homotopic to the composite map

$$g : \partial E = F \times S^1 \to \cup_{i=1}^{r} q_i \times S^1 \to \Gamma$$

such that the first map F × S^1 → $\cup_{i=1}^{r} q_i$ × S1 is induced from the constant map $F \to \cup_{i=1}^{r}\{q_i\}$and the second map $\cup_{i=1}^{r} q_i \times S^1 \to K$is defined by the map f . By using a boundary collar of ∂E in E, assume that the piecewise-linear map f : (E, x_0) → (Γ, x_0) defines the map g : ∂E → Γ. For a non-vertex point p_i of C_i, the preimage $(f)^{-1}(p_i)$ is a bi-collard compact oriented proper piecewise-linear 3-manifold in E. Let V_i be the connected component meeting C_i at the point p_i in E. The boundary ∂Vi is the disjoint union P_i(F) of m_{ij} parallel copies $m_{ij}F_j$ of F_j × 1 for all j (j = 1, 2, . . . , r) in S^4, where mii is an odd integer and m_{ij} with i /= j is an even integers. Let P (F) = ∪r P_i(F) be the surface-link in S^4. Let hi be a disjoint 1-handle system on P_i(F) embedded in V_i such that the surface P_i(F ; h_i) obtained from P_i(F) by surgery along h_i is connected and the genus of P_i(F;hi) is equal to the total genus of P_i(F). Assume that one copy of the parallel $m_{ii}F_i$ of F_i is identified with F_i and just one 1-handle of hi attaches to Fi. Let P (F ; h) = $\cup_{i=1}^{r} P_i(F)$ (F;hi) be a surface-link in S^4. By further taking a disjoint 1-handle system h′ on P_i(F,h_i) embedded in V_i, the closed surface Pi(F ; hi, h′) obtained from P_i(F ; h_i) by surgery along h′ bounds a handlebody in V_i, so that the surface-link P (F ; h, h′) = $\cup_{i=1}^{r} P_i$ (F;h_i,h′) is a trivial surface-link in S^4. Since the compact 4-manifold E′ obtained from E by splitting along $\cup_{ri=1}$ Vi is simply connected, the 1-handle system h′ = $\cup_{i=1}^{r} h'_i$ is a trivial 1-handle system on the surface-link P (F;h) in S^4, [3,28]. Thus, the surface-link P (F ; h) is a trivial surface-link in S^4, [1,2]. The proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1

A ribbon surface-link F is obtained from a trivial S^2-link O in S^4 by surgery along a disjoint 1-handle system h^O on O, so that the surface-link P(F) of a free a ribbon surface-link obtained from a trivial S^2-link P(O) in S^4 by surgery along a disjoint 1-handle system P(h^O) on P (O). Let VP(F) is a SUPH system for the ribbon surface-link P(F) in S^4, namely a multi-punctured handlebody system VP(F) in S^4 such that ∂V P (F) = P (F) ∪ P (O), [22]. Actually, consider the SUPH system VP(F) obtained from the collar P (O) × [0, 1] of O in S^4 by attaching the 1-handle system P (h^O) on P (O) × 0 = P (O). The 1-handle system h = $\cup_{i=1}^{r} h_i$ on P (F) and the SUPH system VP(F) construct a SUPH system VP (F) ∪ h for the trivial surface-link P (F;h). with ∂(VP (F) ∪ h) = P (F;h) ∪ P (O). A spin loop basis (ℓ, ℓ′) for P(F) is the system consisting of a spin loop basis of every component of P (F) where the spin loop system ℓ belongs to a meridian system of the 1-handle system h^O. This system (ℓ,ℓ′) is a spin loop basis of the trivial ribbon surface- link P(F;h). Equivalent ribbon surface-links are faithfully equivalent and they are moved into each other by the moves M_0, M_1, M_2, [30]. This means that there is an orientation-preserving diffeomorphism f of S^4 sending the SUPH system VP(F) ∪ h for P (F;h) to a standard multi-punctured handlebody system W in S^4. By a choice of f , the system (f (ℓ), f (ℓ′)) is a meridian-longitude pair system of the standard multi-punctured handlebody system W in S^4, [26,1]. The loop system f (ℓ′) bounds a disjoint disk system delta δ′ in S^4 with δ′ ∩ W = f (ℓ′), so that the loop system ℓ′ bounds a disjoint disk system D′ = f^{-1}(δ′) in S^4 with D′ ∩ (VP(F) ∪ h) = ℓ′. The loop system ℓ belongs to a meridian system of the 1-handle system P (h^O) and hence bounds a sub-system D of the meridian disk system P(h^O). Thus, it is shown that there is an O2-handle basis (D × I, D′ × I) on P (F) in S^4, whose sub-system to F gives an O2-handle basis on F in S^4. This completes the proof of Theorem 1.1.

The proof of Theorem 1.3 is done as follows.

Proof of Theorem 1.3

The surface-link P(F;h,h′) bounds a disjoint handlebody system V = $\cup_{i=1}^{r} V_i$ in S^4. Let d_i be a meridian disk of the 1-handle h_i, and d = $\cup_{i=1}^{r} d_i$ a meridian disk system of h. Since h′ is a trivial 1-handle system on P (F ; h), there is a disjoint handlebody system U in S^4 with ∂U = P (F ; h) extending the handlebody system V by the uniqueness of an O2-handle pair, [1,2,22]. Then d ∩ U = ∂d. Let (ℓ, ℓ′) be a spin loop basis for P (F ; h) given by a spin loop basis of every component of P (F). Then there is an orientation-preserving diffeomorphism f of S4 sending the handlebody system U to a standard handlebody system W in S^4 such that the spin loop basis (f (ℓ), f (ℓ′)) of W is a meridian-longitude pair system of W, [1,31]. Hence the spin loop basis (f (ℓ), f (ℓ′)) of W bounds a core disk-pair system (δ, δ′) of an O2-handle basis (δ × I, δ′ × I) of the trivial surface-link ∂W in S^4, where δ denotes a meridian disk system of W . This means that the spin loop basis(ℓ, ℓ′) of P (F ; h) bounds the core disk-pair system (D, D′) of the O2-handle basis (D×I, D′ ×I) = (f $^{-1}$(δ)×I, f $^{-1}$(δ′)×I) on P (F ; h) in S^4. The intersection d∩D = ∅ since D ⊂ U. In general, the disk system d meets the disk system D′ transversely with finite interior points in S^4. There is a technique to eliminate the double point system d ∩ D′ by using the O2-handle system (D × I, D′ ×I), [2]. This elimination is actually done by an iteration of the following operation where the 2-handle system D × I on P (F ; h) as a 1-handle system on the surface-link P (F ; h)(D × I):

Finger Move Canceling Operation. Replace the disk system d with a disk system *d′* obtained from *d* and a trivial 2-sphere o linking around a 1-handle in the “1-handle system *D × I*”.

Assume that a disk system d^* obtained from *d* by an iteration of Finger Move Canceling Operation is disjoint from the O2-handle basis (D × I, D′ × I) on P (F ; h) in S^4. Since the loop system ∂d^* = ∂d bounds a disk system d_U in U, consider the 2-sphere system $d_U \cup d^*$ with components denoted by K_i (i = 1, 2, . . . , r). For the surface-link F = $\cup F_i$, this construction can be interpreted as follows: Namely, there is a ribbon surface-link $F^R = \cup_{i=1}^{r} F_i^R$ such that the component F_i of F is the local connected sum $K_i \# F_i^R$ of the S^2-knot K_i and F_i^R for every i (i = 1, 2, . . . , r), where the local connected sums $K_i \# F_i^R$ (i = 1, 2, . . . , r) are connected sums made in disjoint 4-balls B_i (i = 1, 2, . . . , r) in S^4 such that the intersection $F_i^R \cap B_i$ is a trivial proper 2-disk in B_i and $K_i \subset B_i$. The existence of such a 4-ball system B_i (i = 1, 2, . . . , r) is guaranteed by the existence of the O2-handle basis (D × I, D′ × I) on P (F ; h). Let $\tilde{E}$(F) be the maximal free abelian covering of E(F). The fundamental group $\pi_1(\tilde{E}(F), \tilde{x}_0)$ (with $\tilde{x}_0$ a base point lifting x_0) is a free group (since $\pi_1(E(F), x_0)$ is a free group) and contains copies of $\pi_1(\tilde{E}(K_i), \tilde{x}_0)$ (i = 1, 2, . . . , r) as free product summands. Hence the fundamental groups $\pi_1(E(K_i), \tilde{x}_0)$ (i = 1, 2, . . . , r) are free subgroups of $\pi_1(E(F), x_0)$. Since $H_1(E(K_i); Z)$ (i = 1, 2, . . . , r) are infinite cyclic groups, the fundamental groups $\pi_1(E(K_i); x_0)$ (i = 1, 2, . . . , r) are infinite cyclic groups. Then the S^2-knots K_i (i = 1, 2, . . . , r) are trivial S^2-knots, [1,2]. Hence F is equivalent to the ribbon surface-link F^R. This completes the proof of Theorem 1.3.

Acknowledgements

The author has tried to find a simple algebraic method to prove Corollary 1.2, but failed. He is grateful to Igor Mineyev for asking questions to this algebraic method. This work was partly supported by JSPS KAKENHI Grant Number JP21H00978 and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

References

1. Kawauchi, A. (2021). Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link. Topology and its Applications, 301, 107522.
2. Kawauchi, A. (2023). Uniqueness of an orthogonal 2-handle pair on a surface-link. Contemporary mathematics, Universal Wiser Publisher, 4, 182-188.
3. Kawauchi, A. (2023). Triviality of a surface-link with meridian-based free fundamental group. arXiv preprint arXiv:1804.04269.
4. Kōbe Daigaku. Sūgakuka. (1983). Mathematics Seminar Notes (Vol. 11). Kobe University, Department of Mathematics.
5. Yajima, T. (1962). On the fundamental groups of knotted 2-manifolds in the 4-space. Journal of Mathematics, Osaka City University, 13, 63-71.
6. Yajima, T. (1964). On simply knotted spheres in R^4, Osaka Journal of Mathematics, 1, 133-152.
7. Yanagawa, T. (1969). On ribbon 2-knots. The 3-manifold bounded by the 2-knots. Osaka Journal of Mathematics, 6, 447-164.
8. Kawauchi, A. (2024). Classical Poincarè {\'e} conjecture via 4D topology. Journal of Mathematical Techniques and Computational Mathematics, 3(4), 1-7.
9. Perelman, G. (2003). Ricci flow with surgery on three-manifolds. arXiv preprint math/0303109.
10. Poincaré, H. (1900). Second complément à l'analysis situs. Proceedings of the London Mathematical Society, 1(1), 277-308.
11. Poincaré, H. (1904). Cinquième complément à l'analysis situs. Rendiconti del Circolo Matematico di Palermo (1884-1940), 18(1), 45-110.
12. Howie, J. (1983). Some remarks on a problem of JHC Whitehead. Topology, 22(4), 475-485.
13. Kawauchi, A. (2024). Ribbonness of Kervaire's sphere-link in homotopy 4-sphere and its consequences to 2-complexes. Journal of Mathematical Techniques and Computational Mathematics, 3(4), 1-8.
14. Kawauchi, A. (2024). Whitehead aspherical conjecture via ribbon sphere-link. Journal of Mathematical Techniques and Computational Mathematics, 3(5), 1-10.
15. Whitehead, J. H. C. (1941). On adding relations to homotopy groups. Annals of Mathematics, 42(2), 409-428.
16. González-Acuna, F., & Ramirez, A. (2006). A Knot-Theoretic Equivalent of the Kervaire Conjecture. Journal of Knot Theory and Its Ramifications, 15(4), 471-478.
17. Kawauchi, A. (2024). Kervaire conjecture on weight of group via fundamental group of ribbon sphere-link. Journal of Mathematical Techniques and Computational Mathematics, 3(4), 1-3.
18. Kervaire, M. A. (1965). On higher dimensional knots. Differential and combinatorial topology, Princeton Univ. Press. 105-119.
19. Klyachko, A. A. (1993). A funny property of sphere and equations over groups. Communications in Algebra, 21(7), 2555-2575.
20. Magnus, W., & Karrass, A., & Solitar, D. (2004). Combinatorial group theory: Presentations of groups in terms of generators and relations. Courier Corporation.
21. Kawauchi, A. (2023). Smooth homotopy 4-sphere. WSEA Transactions on Mathematics 22, 690-701.
22. Kawauchi, A. (2025). Ribbonness of a stable-ribbon surface-link, II. General case. (MDPI) Mathematics, 13(3), 402, 1-11.
23. Kawauchi, A. (2024). Another proof of free ribbon lemma. Journal of Mathematical Techniques and Computational Mathematics, 3(9), 01-03.
24. Kawauchi, A. (2006). On the surface-link groups. Intelligence of low dimensional topology, 157-164.
25. Kawauchi, A. (2008). The first Alexander Z[Z]-modules of surface-links and of virtual links. The geometry and Topology Monographs, The Zieschang Gedenkschrift, 14, 353-371.
26. Yajima, T. (1970). Wirtinger presentations of knot groups. Proceedings of the Japan Academy, 46(9), 997-1000.v
27. Kamada, S. (2001). Wirtinger presentations for higher dimensional manifold knots obtained from diagrams. Fundamenta Mathematicae, 168, 105-112.
28. Hosokawa, F., & Kawauchi, A. (1979). Proposals for unknotted surfaces in four-spaces.

29. Spanier, E. H., & Spanier, E. H. (1989). Algebraic topology. Springer Science & Business Media.

30. Kawauchi, A. (2018). Faithful equivalence of equivalent ribbon surface-links. Journal of Knot Theory and Its Ramifications, 27(11), 1843003.

31. Hirose, S. (2002). On diffeomorphisms over surfaces trivially embedded in the 4–sphere. Algebraic & Geometric Topology, 2(2), 791-824.

32. Hillman, J. A., & Kawauchi, A. (1995). Unknotting orientable surfaces in the 4-sphere. Journal of Knot Theory and Its Ramifications, 4(02), 213-224.