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# Schur's Orthogonality Relations for the Free Group

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### Abstarct

*We prove an explicit convergence of suitably normalized integrals on balls where the integrand is the product of coefficients of the quasi-regular representation of the finitely generated free group. This result follows from the fact that the quasi-regular representation of the free group is c-tempered in the sens of Kazhdan and Yom Din [KYD22, def 2.1]. The convergence can be summed up by the formula:*

$$\lim_{n\to\infty}\frac{1}{n^3}\int_{B_n}\langle\pi(g)\psi_1,\psi_2\rangle\overline{\langle\pi(g)\psi_3,\psi_4\rangle}d\mu(g)=\frac{3q(q+1)}{(q-1)^2}\langle\psi_1,\psi_3\rangle\overline{\langle\psi_2,\psi_4\rangle}$$

where q +1 is twice the rank of the free group, Bn is the ball of radius n with respect to the canonical word metric and $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ are square summable functions on the boundary of the free group.

**Keywords and Phrases:** Free Group, Unitary Representation, Quasi-Regular Representation, Schur's Orthogonality Relations, C-Tempered, Harish-Chandra Function.

## 1. Introduction

Given an irreducible unitary representation of a compact group ρ : G → U (V), we know (see for instance [BBP23]) that the Hilbert space V is finite dimensional (say d := dimC(V)) and, for all $v_1$, $v_2$, $v_3$ and $v_4$ in $V$, we have:

$$\int_G\langle\rho(g)v_1,v_2\rangle\overline{\langle\rho(g)v_3,v_4\rangle}dg=\frac{1}{d}\langle v_1,v_3\rangle\overline{\langle v_2,v_4\rangle}$$

where *dg* denotes the normalised Haar measure on *G*.

This formula (Schur's orthogonality relations) can be seen as a generalisation of the fact that characters of finite groups are unitary in $l^2(G, C)$ but does not make sense when G is not compact and the coefficients not square summable.

In this paper, we compute an equivalent asymptotic formula for the boundary representation of the free group.

In [BG16, First theorem], one can find similar results for Gromov hyperbolic groups when the metric is non arithmetic.

## 1.1 Settings and Notations

Let $G$ be the free group with N generators, X the Cayley graph associated to right multiplication in $G$ and $x_0$ a base point in $X$. Then $X$ is an homogeneous tree of degree $2N$ =: q+1 equipped with the unique distance d which gives the value 1 to any pair of adjacent vertices (see figure 1 for an example where $N = 2$ and $x_0$ = e the neutral element of $G$). We denote by $[x, y]$ the unique geodesic joining $x$ to $y$ in $X$, $S_k := S(x_0, k) := \{x < X||x| := d(x_0, x) = k\}$ the sphere centered at $x_0$ with radius $k < N$ (for an element g of $G$, we write also g < $S_k$ whenever $gx_0 < S(x_0, k)$ as an element of $X$). We also denote by $B_n$:= B(x0, n) the ball with radius n < **N**.

A point $\omega$ of the boundary Ω can be seen as a direction to infinity or, more precisely, as an infinite geodesics $[x_0, \omega)$ starting at $x_0$. We equip Ω with a topological structure declaring its basis of open sets to be all the shadows $\Omega_x = \{\omega < \Omega|[x_0, x]$ Ă $[x_0, \omega)\}$, where x is in X, which makes Ω a compact topological space (for more details, see the introduction of [KS92] where Kuhn embeds X $\bigsqcup$ Ω in a cartesian product of compact spaces).

We also equip Ω with a Borel probability measure $v$ which satisfy:

$$\forall x \in X, \nu(\Omega_x) = \frac{1}{|S_{|x|}|} = \frac{1}{(q+1)q^{|x|-1}}.$$

The isometric left action of $G$ on $X$ clearly extends to a left action on Ω and one can show that $v$ is quasi-invariant under this action

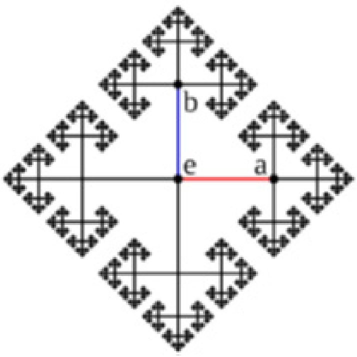


**Figure 1:** Example for $N = 2$ and $x_0$ = e

[BOU95, Corollary 2.6.3] where Bourdon shows that $G$ acts by conformal maps on Ω).

In particular, for all $g$ in $G$, $g_* \nu << \nu$. One can show that $\frac{dg_*\nu}{d\nu}$ (ω) = P $(g^{-1}, \omega)$ := $q^{\beta\omega(x0,gx0)}$ where $\beta$ is the Busemann function (see also [BOU95]).

Let $H$ be the Hilbert space $L^2(\Omega, \nu, C)$. We define the unitary representation π: $G \rightarrow U(H)$ by:

$$\forall g \in G, \psi \in \mathcal{H}, \omega \in \Omega, \pi(g)\psi(\omega) := P(g^{-1}, \omega)^{\frac{1}{2}}\psi(g^{-1}\omega).$$

It is well known $\pi$ is irreducible (see for example [BL17, thm 1.2]). In particular, by Schur's lemma (1)

$Hom_G(\pi, \pi) = \{T < B(H)$ such that $T \circ \pi(g) = \pi(g) \circ T, ðg < G\} = \mathbf{C}Id_H$.

### 1.1 Organisation of the Paper

In section 2 we compute the values of the Harish-Chandra function and prove that it is spherical using partitions of the boundary where the Busemann function is constant. In section 3, we show (using results from [BL17]) that the representation is c-tempered in the sense of [KYD22] (using results from [Kuh94] and [Haa78] about the extension of our representation and the regular one). To conclude, in section 4, we adapt and detail the proof one can find in [KYD22] to obtain our asymptotic orthogonality relations by showing some intermediate results based on functional analysis from [RS81] and explain how we obtain our initially mentioned formula.

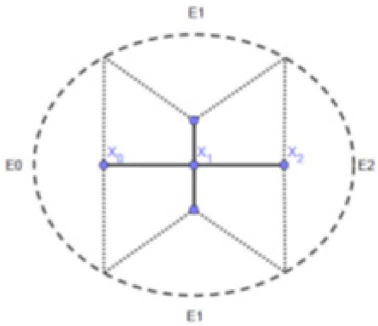


**Figure 2**: Example with $[x_0, x] = (x_0, x_1, x_2)$ illustrating Lemma 2.1 where the family $(E_0, E_1, E_2)$ is a partition of the boundary.

## 2. Computing the Harish-Chandra Function

### 2.1 Partitions where the Busemann Function is Constant

Lemma 2.1

Let $x < S_n$ such that $[x_0, x] = (x_0, x_1, \ldots, x = x_n)$. One defines the following sets:

$$E_k(x) := \begin{cases} \Omega_x & \text{if } k = n \\ \Omega_{x_k} - \Omega_{x_{k+1}} & \text{if } 0 \leqslant k < n \end{cases}$$

Then $\{E_0(x), \ldots, E_n(x)\}$ is a partition of $\Omega$ (see Figure 2 for an example where $n = 2$).

Moreover, $\omega \to \beta_\omega(x_0, x)$ is constantly equal to $2k - n$ on $E_k(x)$.

One computes:

$$\nu(E_k(x)) = \begin{cases} \nu(\Omega_x) = \frac{1}{(q+1)q^{n-1}} & (\text{if } k = n) \\ \nu(\Omega_{x_k}) - \nu(\Omega_{x_{k+1}}) = \frac{q-1}{(q+1)q^k} & (\text{if } k \in [1, n-1]) \\ 1 - \nu(\Omega_{x_1}) = \frac{q}{q+1} & (\text{if } k = 0) \end{cases}$$

### 2.2 Computation: The Harish-Chandra Function is Spherical

Here we define $\Xi$ and show that it is constant on Sn by computing its value.

$$\begin{array}{rccc} \Xi : & G & \longrightarrow & \mathbf{R}^{+*} \\ & g & \longmapsto & \langle \pi(g)1_\Omega, 1_\Omega \rangle \end{array}$$

Note that Ξ(0) := Ξ(e) = 1

Now, let n ≥1, and $g$ such that $g < S_n$ and compute

$$\begin{aligned} \Xi(g) &= \int_\Omega \pi(g)1_\Omega \overline{1_\Omega} d\nu = \int_\Omega P(g^{-1}, \omega)^{\frac{1}{2}} d\nu(\omega) = \sum_{k=0}^{n} \int_{E_k(gx_0)} q^{\frac{\beta_\omega(x_0, gx_0)}{2}} d\nu(\omega) \\ &= \sum_{k=0}^{n} q^{\frac{2k-n}{2}} \nu(E_k(gx_0)) \qquad \text{(by Lemma 2.1)} \\ &= q^{-\frac{n}{2}} \frac{q}{q+1} + \sum_{k=1}^{n-1} q^{\frac{2k-n}{2}} \frac{q-1}{(q+1)q^k} + q^{\frac{n}{2}} \frac{1}{(q+1)q^{n-1}} \qquad \text{(from Equation 2)} \\ &= \left[1 + (\frac{q-1}{q+1})n\right] q^{-\frac{n}{2}}. \end{aligned}$$

So, Ξ is spherical and we can define:

Ξ : N → R→*

n'→ Ξ(g) (where g is any element of $S_n$)

## 1. *π* is C-Tempered

Here we will prove that π is c-tempered, in the sense of Kazhdan and Yom Din in section 2 «Notion of c-temperedness»[KYD22, def 2.1].

For all subset $L \subset G$ and all $\psi_1$, $\psi_2$ < H, one can define the quantity:

$$M_{\psi_1,\psi_2}(L) := \int_L |\langle \pi(g)\psi_1, \psi_2 \rangle|^2 d\mu(g)$$

In particular, we have

$$M_{1_\Omega,1_\Omega}(L) = \int_L \Xi(g)^2 d\mu(g)$$

Using the spherical property of Ξ, one computes:

$$M_{1_\Omega,1_\Omega}(S_k) = \frac{q+1}{q}\left(1 + \frac{q-1}{q+1}k\right)^2 \quad (k > 0).$$

(3) and (4)

$$
\begin{aligned}
M_{1_\Omega,1_\Omega}(B_n) &= \sum_{k=0}^{n} M_{1_\Omega,1_\Omega}(S_k) \\
&= 1 + \frac{q+1}{q}\left[\frac{n(n+1)(2n+1)}{6}\left(\frac{q-1}{q+1}\right)^2 + n(n+1)\frac{q-1}{q+1} + n\right] \\
&\underset{n\to\infty}{\sim} \frac{n^3}{K} \text{ where } K = \frac{3q(q+1)}{(q-1)^2}.
\end{aligned}
$$

Considering the sequence $\{B_n\}_{n\in N}$ of balls in G and our unit vector 1Ω < H. Then if the two following conditions are satisfied:

(5)

One says that π is c-tempered with Følner sequence $\{B_n\}$nPN (see [KYD22, def 2.1]).

### Remark 3.1

The condition (6) is equivalent to the second condition in the definition [KYD22, def 2.1] because G is discret, therefore its compacts are the finite sets.

### Lemma 3.2

π satisfies the first condition (5).

### Proof

By [Kuh94], we know that π is weakly contained in the regular representation $\pi_{reg}$. In particular, the extensions on $l^1(G)$ of these representations:

$$
\rho^{ext}\left(\sum_{g\in G} a_g\delta_g\right) = \int_G a_g\rho(g)dg = \sum_{g\in G} a_g\rho(g)
$$

(where ρ < {π, πreg}) satisfy:

$$
\forall f \in l^1(G), \quad \|\pi^{ext}(f)\|_{op} \leqslant \|\pi^{ext}_{reg}(f)\|_{op}.
$$

Moreover, by [Haa78], we have:

$$
\|\pi^{ext}_{reg}(f)\|_{op} \leqslant \sum_{k=0}^{\infty}(k+1)\|f1_{S_k}\|_2.
$$

Consider the sequence of $l^2(G)$ functions $\{f_k\}_k$ defined as

$$
f_k(g) := 1_{S_k}(g)\overline{\langle\pi(g)\psi_1, \psi_2\rangle} \text{ where } \psi_1, \psi_2 \in \mathcal{H}.
$$

One has, fixing arbitrary unitary $\psi_1$, $\psi_2$ < H:

$$
\begin{aligned}
0 < M_{\psi_1,\psi_2}(S_k) &= \langle\pi^{ext}(f_k)\psi_1, \psi_2\rangle \\
&\leqslant \|\pi^{ext}(f_k)\|_{op}\|\psi_1\|_2\|\psi_2\|_2 \leqslant \|\pi^{ext}_{reg}(f_k)\|_{op} \\
&\leqslant (k+1)\|f_k\|_2.
\end{aligned}
$$

and since $\|f_k\|_2 = M_{\psi_1,\psi_2}(S_k)^{\frac{1}{2}}$, we obtain

$$M_{\psi_1,\psi_2}(S_k) \leqslant (k+1)^2.$$

So, by equation (3),

$$\frac{M_{\psi_1,\psi_2}(S_k)}{M_{1_\Omega,1_\Omega}(S_k)} \leqslant \frac{q}{q+1}u_k^2 \quad \left(\text{for } u_k := \frac{1+k}{1+\frac{q-1}{q+1}k}\right)$$

But one can easily check that $(u_k)_{k\in N}$ is bounded. So there is a $C < R^+$ (which does not depend on k) such that:

$$\forall \psi_1, \psi_2 \in \mathcal{H}, \quad M_{\psi_1,\psi_2}(S_k) \leqslant CM_{1_\Omega,1_\Omega}(S_k). \tag{7}$$

Hence,

$$M_{\psi_1,\psi_2}(B_n) = \sum_{k=0}^{n} M_{\psi_1,\psi_2}(S_k) \leqslant \sum_{k=0}^{n} CM_{1_\Omega,1_\Omega}(S_k) = CM_{1_\Omega,1_\Omega}(B_n)$$

**Lemma 3.3**

π satisfies the second condition (6).

**Proof**

Let k := |g| + |h|

One can easily show that:

$$B_n \Delta h^{-1} B_n g \subset B_{n+k} - B_{n-k}$$

and

$$M_{\psi_1,\psi_2}(B_n \Delta h^{-1} B_n g) \leqslant M_{\psi_1,\psi_2}(B_{n+k} - B_{n-k}) = \sum_{j=n-k+1}^{n+k} M_{\psi_1,\psi_2}(S^j)$$

$$\overset{\text{(by 7)}}{\leqslant} C \sum_{j=n-k+1}^{n+k} M_{1_\Omega,1_\Omega}(S^j) = C\big(M_{1_\Omega,1_\Omega}(B_{n+k}) - M_{1_\Omega,1_\Omega}(B_{n-k})\big)$$

Recalling the computation done in (4) which gives:

$$M_{1_\Omega,1_\Omega}(B_m) \underset{m\to\infty}{\sim} \frac{m^3}{K}$$

So,

$$\frac{M_{\psi_1,\psi_2}(B_n\Delta h^{-1}B_ng)}{M_{1_\Omega,1_\Omega}(B_n)} \leqslant C\frac{M_{1_\Omega,1_\Omega}(B_{n+k})-M_{1_\Omega,1_\Omega}(B_{n-k})}{M_{1_\Omega,1_\Omega}(B_n)} \underset{n\to\infty}{\sim} C\frac{(n+k)^3-(n-k)^3}{n^3}.$$

The left hand side converging to zero since the degree 3 coefficient of the numerator vanishes.

## 4. Asvmptotic Schur's Orthogonalitv Relations for $\pi$

Now that we have the conditions (5) and (6) for our representation, we can detail the proof of proposition 2.3 found in [KYD22]. Namely, in our case, for all $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ < H:

$$\lim_{n\to\infty}\frac{\int_{B_n}\langle\pi(g)\psi_1,\psi_2\rangle\overline{\langle\pi(g)\psi_3,\psi_4\rangle}d\mu(g)}{M_{1_\Omega,1_\Omega}(B_n)} = \langle\psi_1,\psi_3\rangle\overline{\langle\psi_2,\psi_4\rangle}. \quad (8)$$

We denote by $\overline{\mathcal{H}}$ the conjugate of our vector space $\mathcal{H}$. This allows us to see any sesquilinear form of $\mathcal{H}$ (like $\langle\pi(g)\cdot,\cdot\rangle, \forall g\in G)$ to be a bilinear one on $\mathcal{H}\times\overline{\mathcal{H}}$.

### Lemma 4.1

Let $B$: $\mathcal{H}\times\overline{\mathcal{H}}\to$ C be a bounded bilinear form such that $B(\pi(g)\psi_1, \pi(g)\psi_3) = B(\psi_1, \psi_3)$
$\forall g\in G \;\; and \;\; \psi_1,\psi_3\in\mathcal{H}.$

Then *B € C(·, ·).*
*In other words, there is a constant λ in C such that B = λ(·, ·).*

### Proof

For all ψ*1* in H, Riesz lemma gives us an element T ($\psi_1$) such that $B(\psi_1, \cdot) = (T(\psi_1), \cdot)$.
This defines a map T: *H* → C which is linear and bounded. Indeed:

$$\|\underbrace{T(\lambda\psi_1+\Psi_1)-\lambda T(\psi_1)-T(\Psi_1)}_{=:\psi}\|^2 = B(\lambda\psi_1+\Psi_1,\psi)-\lambda B(\psi_1,\psi)-B(\Psi_1,\psi)=0.$$

and,

$$\|T(\psi_1)\|\leqslant\|B\|\|\psi_1\|,$$

Moreover, T is an intertwining operator since:

$$\|\underbrace{(T\circ\pi(g)-\pi(g)\circ T)\psi_1}_{=:\psi}\|^2 = B(\pi(g)\psi_1,\psi)-B(\psi_1,\pi(g^{-1})\psi)=0$$

Hence, by irreducibility of π and the application of Schur's Lemma mentioned in (1) implies that T < CIdH and B = (T (·), ·) < C(·, ·).

### Remark 4.2

If D: $\overline{\mathcal{H}} \times \mathcal{H}$→ C is a bounded bilinear form such that $\forall g \in G$ and $\psi_2$, $\psi_4$ < H, D(π(g)$\psi_2$, π(g)$\psi_4$) = B($\psi_2$, $\psi_4$).

Then, composing it with the flip operator F which swap the coordinates, we obtain that B:= D◦F satisfies the conditions of the previous lemma (4.1). So D ∘ F < C(·, ·), that is to say D < C(·, ·).

One last simple lemma (about convergence in C) before proving the equation (8) mentioned at the beginning of the section:

### Lemma 4.3

Let $(u_n)_{n€N}$ € $C^N$ and *l € C*. Then

$\lim_{n\to\infty} u_n = l$ is equivalent to the following condition:

For all subsequences $(u_{\alpha(n)})_{n€N}$, there is a further subsequence $(u_{\alpha˚\gamma(n)})_{n€N}$

such that $\lim_{n\to\infty} u_{\alpha\circ\gamma(n)} = l.$

### Theorem 4.4

Given any $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$€H:

$$\lim_{n\to\infty} \frac{\int_{B_n} \langle \pi(g)\psi_1, \psi_2\rangle \overline{\langle \pi(g)\psi_3, \psi_4\rangle} d\mu(g)}{M_{1_\Omega,1_\Omega}(B_n)} = \langle \psi_1, \psi_3\rangle \overline{\langle \psi_2, \psi_4\rangle}.$$

### Proof

Define $\forall n \in \mathbf{N}$

$$S_n : \mathcal{H} \times \overline{\mathcal{H}} \to L^2(G, \mu), \text{ by}$$

$$S_n(\psi_1, \psi_2)(g) := 1_{B_n} \frac{\langle \pi(g)\psi_1, \psi_2\rangle}{M_{1_\Omega,1_\Omega}(B_n)^{\frac{1}{2}}}.$$

Then Sn is clearly bilinear and, since π is c-tempered, we have by the first condition of c-temperedness (5),

$$\limsup_{n\to\infty} \int_\omega |S_n(\psi_1, \psi_2)(g)|^2 d\mu(g) = \limsup_{n\to\infty} \frac{M_{\psi_1,\psi_2}(B_n)}{M_{1_\Omega,1_\Omega}(B_n)} < \infty$$

This shows that *(Sn($\psi_1$, $\psi_2$))nPN* bounded in $L^2$(G) and, by the Banach- Steinhaus theorem,

$$\exists C \in \mathbf{R}^+ \text{ such that } \|S_n\|^2 \leqslant C. \qquad (9)$$

Now, $\forall n \in \mathbf{N}$, one can define the quadrilinear form

$$Q_n : \mathcal{H} \times \overline{\mathcal{H}} \times \overline{\mathcal{H}} \times \mathcal{H} \to \mathbf{C} \text{ as:}$$
$$Q_n(\psi_1, \psi_2, \psi_3, \psi_4) := \langle S_n(\psi_1, \psi_2), S_n(\psi_3, \psi_4)\rangle.$$

(Qn)nPN is also uniformly bounded, since, using Cauchy-Schwartz in- equality and equation (9), we see, for unitary $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$:

$$\|Q_n(\psi_1, \psi_2, \psi_3, \psi_4)\| \leqslant \|S_n(\psi_1, \psi_2)\| \|S_n(\psi_3, \psi_4)\| \leqslant C$$

Next we will show that, $\forall g, h \in G$, we have:

$$\lim_{n \to \infty} \Big( Q_n(\pi(g)\psi_1, \pi(h)\psi_2, \pi(g)\psi_3, \pi(h)\psi_4) - Q_n(\psi_1, \psi_2, \psi_3, \psi_4) \Big) = 0. \qquad (10)$$

Indeed, one can show that:

$$|Q_n(\pi(g)\psi_1, \pi(h)\psi_2, \pi(g)\psi_3, \pi(h)\psi_4) - Q_n(\psi_1, \psi_2, \psi_3, \psi_4)|$$
$$\leqslant \left( \frac{M_{\psi_1,\psi_2}(B_n \Delta h^{-1} B_n g)}{M_{1_\Omega,1_\Omega}(B_n)} \right)^{\frac{1}{2}} \left( \frac{M_{\psi_3,\psi_4}(B_n \Delta h^{-1} B_n g)}{M_{1_\Omega,1_\Omega}(B_n)} \right)^{\frac{1}{2}}$$

which tends to 0 when n goes to infinity since π is c-tempered (see condition 2 of c-temperedness (6)).

Now, One can consider any subsequence $(Q_{\alpha(n)})_{n \in N}$ and an application of the Banach-Alaoglu theorem in $\overline{B_{\|\|}(0, C)}$ gives us a further subsequence $(Q_{\alpha\circ\gamma(n)})_{n \in N}$ which converges point-wise to some $Q$ in $\overline{B_{\|\|}(0, C)}$:

(11) $$\forall \psi_1, \psi_2, \psi_3, \psi_4, \quad Q(\psi_1, \psi_2, \psi_3, \psi_4) = \lim_{n \to \infty} Q_{\alpha\circ\gamma(n)}(\psi_1, \psi_2, \psi_3, \psi_4) \text{ in } \mathbf{C}.$$

Now, by (10), we have:
$Q(\pi(g)\psi_1, \pi(h)\psi_2, \pi(g)\psi_3, \pi(h)\psi_4) = Q(\psi_1, \psi_2, \psi_3, \psi_4)$
and $Q$ is also quadrilinear.
In particular, if we fix (ψ2, ψ4) < H × H and consider the map
$Q(\cdot, \psi_2, \cdot, \psi_4)$ (which satisfies the conditions of lemma 4.1).

We know that, $\exists \lambda_{\psi_2,\psi_4} \in \mathbf{C}$ such that:
$Q(\cdot, \psi_2, \cdot, \psi_4) = \lambda_{\psi_2,\psi_4} (\cdot, \cdot)$.
Similarly, fixing $(\psi_1, \psi_3) \in \mathcal{H} \times \overline{\mathcal{H}}$ and by the remark 4.2 following the mentioned lemma, we have $\exists \lambda^{\psi_1,\psi_3} \in \mathbf{C}$ such that:

$$Q(\psi_1, \cdot, \psi_3, \cdot) = \lambda^{\psi_1,\psi_3} \overline{\langle \cdot, \cdot \rangle}.$$

Next, using the definition of these coefficient one calculate $\lambda_{1\Omega,1\Omega} = 1$, $\lambda^{\psi 1,\psi 3} = (\psi_1, \psi_3)$ and $Q(\psi_1, \psi_2, \psi_3, \psi_4) = \lambda^{\psi 1,\psi 3} (\psi_2, \psi_4) = (\psi_1, \psi_3)(\psi_2, \psi_4)$. To finish the proof, Let $\psi_1, \psi_2, \psi_3, \psi_4$ € H and define:

$$u_n := \frac{\int_{B_n} \langle \pi(g)\psi_1, \psi_2 \rangle \overline{\langle \pi(g)\psi_3, \psi_4 \rangle} d\mu(g)}{M_{1_\Omega, 1_\Omega}(B_n)}$$

and consider any subsequence $(u_{\alpha(n)})_{n \in N}$.

Since $u_{\alpha(n)} = Q_{\alpha(n)}(\psi_1, \psi_2, {}_{\psi 3,}\ \psi_4)$, has further subsequence converging to

$$\langle \psi_1, \psi_3 \rangle \overline{\langle \psi_2, \psi_4 \rangle}, \text{ we have by the lemma (4.3), } \lim_{n\to\infty} u_n = \langle \psi_1, \psi_3 \rangle \overline{\langle \psi_2, \psi_4 \rangle}.$$

### Remark 4.5

A unitary representation which satisfies these orthogonality relations has to be irreducible.

So, using the equation result (8) of our last theorem, and the computation of $M_{1\Omega,1\Omega}(B_n)$ we did in (4), we obtain the concrete result mentioned at the beginning of this paper:

$$\lim_{n\to\infty} \frac{1}{n^3} \int_{B_n} \langle \pi(g)\psi_1, \psi_2 \rangle \overline{\langle \pi(g)\psi_3, \psi_4 \rangle} d\mu(g) = \frac{3q(q+1)}{(q-1)^2} \langle \psi_1, \psi_3 \rangle \overline{\langle \psi_2, \psi_4 \rangle}.$$

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